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Existence of the primitive Weierstrass gap sequences on curves of genus 9

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Abstract. We show that for any possible Weierstrass gap sequence L on a curve of genus 9 with twice the smallest positive non-gap > the largest gap there exists a pointed non-singular curve (C, P) over an algebraically closed field of characteristic 0 such that the gap sequence at P is L.

Keywords: non-singular curves, gap sequences, toric varieties, trigonal curves.

1. Introduction.

Let C be a complete non-singular irreducible algebraic curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let P be its point. A positive integer γ is called a *gap* at P if there exists a regular 1-form ω on C such that $\mathrm{ord}_P(\omega) = \gamma - 1$. We denote by $L(P)$ the set of gaps at P, which is also called the *gap sequence* at P. Then the cardinality of $L(P)$ is equal to g. Moreover, the complement $H(P)$ of $L(P)$ in the additive semigroup $\mathbb{N}$ of non-negative integers forms a subsemigroup of $\mathbb{N}$.

Conversely, let L be a *gap sequence*, i.e., a finite subset of $\mathbb{N}$ whose complement $H(L) = \mathbb{N}\backslash L$ in $\mathbb{N}$ forms a subsemigroup of $\mathbb{N}$. The cardinality of L is called its *genus*. We say that L is *Weierstrass* if there exists a pointed curve (C, P) such that $L(P) = L$. Buchweitz [**1**] showed that there is a non-Weierstrass gap sequence of genus 16. We are interested in the maximal genus g such that all gap sequences of genus g are Weierstrass. In fact the author (Komeda [**10**]) showed that all gap sequences of genus ≤ 7 are Weierstrass and that all *primitive* gap

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sequences, i.e., twice the smallest positive integer in $H(L) >$ the largest integer in L, of genus 8 are Weierstrass.

In this paper we study primitive gap sequences of genus 9 and show the following:

Main Theorem. *All primitive gap sequences of genus 9 are Weierstrass.*

The following are the main ingredients of the proof of the Main Theorem:

(1) For several gap sequences L we construct affine toric varieties associated with L for applying Corollary 4.9 in Komeda [**7**].

(2) We calculate the dimension of the moduli space of the pointed curves (C, P) of genus 8 with $L(P) = \{1, \ldots, 6, 12, 13\}$ using the method of Stöhr-Viana [**14**] for applying Theorem 5.4 in Eisenbud-Harris [**4**].

2. On primitive gap sequences of genus 9.

For a gap sequence $L = \{l_0 < l_1 < \ldots < l_{g-1}\}$ of genus g, let $M(L)$ be the minimal set of generators for the semigroup $H(L)$. Set

$$\alpha(L) = (\alpha_0(L), \alpha_1(L), \ldots, \alpha_{g-1}(L)),$$

where $\alpha_i(L) = l_i - i - 1$ for any $i = 0, 1, \ldots, g-1$. Moreover, set

$$w(L) = \sum_{i=0}^{g-1} \alpha_i(L),$$

which is called the *weight* of L. We denote by $a(L)$ the smallest positive integer in $H(L)$. Then $2 \leq a(L) \leq g+1$. If $a(L) = 2$, then $L = \{1, 3, \ldots, 2g-1\}$, which is Weierstrass. If $a(L) = 3$ (resp. 4, resp. 5, resp. g), then L is Weierstrass by Maclachlan [**11**] (resp. Komeda [**7**], resp. Komeda [**9**], resp. Pinkham [**13**]). Hence we only consider the cases $6 \leq a(L) \leq g-1$. Eisenbud-Harris [**4**] (resp. Komeda [**8**]) showed that any primitive gap sequence of genus g and weight less than $g-1$ (resp. equal to $g-1$) is Weierstrass. Moreover, any primitive gap sequence L of genus g and weight g with $\alpha(L) = (0^{g-2}, m, n)$ is Weierstrass by Proposition 4.4 in Komeda [**10**]. Kim [**6**] showed that for any gap sequence L with $\alpha(L) = (0^{g-r}, m^r)$ there exists a pointed trigonal curve (C, P) such that $L(P) = L$. Thus to prove that all primitive gap sequences of genus 9 are

Weierstrass it suffices to show that the 7 sequences in the table below are Weierstrass.

	L	$M(L)$	$\alpha(L)$	$w(L)$
(1)	$\{1,2,3,4,5,6,10,12,13\}$	$\{7,8,9,11\}$	$(0^6,3,4^2)$	11
(2)	$\{1,2,3,4,5,6,10,11,13\}$	$\{7,8,9,12\}$	$(0^6,3^2,4)$	10
(3)	$\{1,2,3,4,5,6,9,12,13\}$	$\{7,8,10,11\}$	$(0^6,2,4^2)$	10
(4)	$\{1,2,3,4,5,6,9,11,13\}$	$\{7,8,10,12\}$	$(0^6,2,3,4)$	9
(5)	$\{1,2,3,4,5,6,8,12,13\}$	$\{7,9,10,11,15\}$	$(0^6,1,4^2)$	9
(6)	$\{1,2,3,4,5,6,7,13,15\}$	$\{8,9,10,11,12,14\}$	$(0^7,5,6)$	11
(7)	$\{1,2,3,4,5,6,7,12,15\}$	$\{8,9,10,11,13,14\}$	$(0^7,4,6)$	10

3. The construction of affine toric varieties associated with gap sequences.

To prove that the gap sequences (1),(2),(3) and (4) in the table are Weierstrass we apply Corollary 4.9 in Komeda [**7**] to these cases. Hence we must construct an affine toric variety associated with each gap sequence. First we prepare some notations. Let $\mathbb{Z}$ be the set of integers. For any $i = 1,\dots,n$ we denote by e_i the vector in $\mathbb{Z}^n$ whose i-th component is equal to 1 and whose j-th component is equal to 0 if $j \neq i$. Let L be a gap sequence. Set $M(L) = \{a_1,\dots,a_n\}$. Let $\varphi_L : k[X] = k[X_1,\dots,X_n] \longrightarrow k[H(L)] = k[t^h]_{h\in H(L)}$ be the k-algebra homomorphism defined by sending X_i to t^{a_i} for each $i = 1,\dots,n$. We denote by I_L the ideal Ker φ_L. Moreover, we define a weight on $k[X]$ as follows: For any i, the weighted degree of X_i is a_i and for any non-zero element c of k, the weighted degree of c is zero. For any monomial f in $k[X]$, $w(f)$ denotes the weighted degree of f.

The Case (1). Set $a_1 = 7, a_2 = 8, a_3 = 9, a_4 = 11$. Then we have relations:

$$4a_1 = a_2 + a_3 + a_4, \quad 2a_2 = a_1 + a_3, \quad 2a_3 = a_1 + a_4 \text{ and } 2a_4 = 2a_1 + a_2.$$

Using Lemma 4.12 in Komeda [**7**] the ideal I_L is generated by

$$X_1^4 - X_2X_3X_4, \quad X_2^2 - X_1X_3, \; X_3^2 - X_1X_4 \text{ and } X_4^2 - X_1^2X_2.$$

Set $g_1 = X_1$, $g_2 = X_1$, $g_3 = X_1^2$, $g_4 = X_2$, $g_5 = X_3, g_6 = X_2$, $g_7 = X_4$, $g_8 = X_3$ and $g_9 = X_4$.

Let S be the subsemigroup of $\mathbb{Z}^6$ generated by $b_1, b_2, \ldots, b_9$, where $b_i = e_i$ for $i = 1, \ldots, 6$, $b_7 = e_1 + e_2 + e_3 - e_4 - e_5$, $b_8 = e_4 + e_6 - e_1$ and $b_9 = b_5 + b_8 - b_2 = e_4 + e_5 + e_6 - e_1 - e_2$. To prove that S is saturated it suffices to show that

$$\sum_{i=1}^{9} \mathbb{R}_+ b_i \cap \mathbb{Z}^6 \subseteq \sum_{i=1}^{9} \mathbb{N} b_i = S$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Let

$$p = (p_1, \ldots, p_6) = \sum_{i=1}^{9} m_i b_i \in \mathbb{Z}^6$$

with $m_i \in \mathbb{R}_+$ for all i. Then we may assume that $0 \leq m_i < 1$ for all i. Hence

$$p_1 = m_1 + m_7 - m_8 - m_9 \geq -1, \quad p_2 = m_2 + m_7 - m_9 \geq 0,$$
$$p_3 = m_3 + m_7 \geq 0, \quad p_4 = m_4 - m_7 + m_8 + m_9 \geq 0,$$
$$p_5 = m_5 - m_7 + m_9 \geq 0 \quad \text{and} \quad p_6 = m_6 + m_8 + m_9 \geq 0.$$

It suffices to show that if $p_1 = -1$, then $p \in S$. Then $m_1 + m_7 + 1 = m_8 + m_9$, which implies that $p_4 \geq 1$ and $p_6 \geq 1$. Hence we may assume that $p = (-1, 0, 0, 1, 0, 1)$, which implies that $p = b_8 \in S$. Let

$$\pi \colon k[Y] = k[Y_1, \ldots, Y_9] \longrightarrow k[S] = k[T^s]_{s \in S} \text{ (resp. } \eta \colon k[Y] \longrightarrow k[X])$$

be the k-algebra homomorphism defined by $\pi(Y_i) = T^{b_i}$ (resp. $\eta(Y_i) = g_i$) where for any $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ we denote by T^p the monomial $t_1^{p_1} \cdots t_n^{p_n}$. Now the k-algebra homomorphism

$$\zeta : k[\mathbb{N}^6] = k[t_1, \ldots, t_6] \longrightarrow k[H(L)]$$

defined by $\zeta(t_i) = t^{w(g_i)}$ extends to $\zeta' : k[S] \longrightarrow k[H(L)]$, because

$$w(g_1 g_2 g_3 g_4^{-1} g_5^{-1}) = w(g_7), \quad w(g_4 g_6 g_1^{-1}) = w(g_8) \text{ and}$$
$$w(g_4 g_5 g_6 g_1^{-1} g_2^{-1}) = w(g_9).$$

Then $\varphi_L \circ \eta = \zeta' \circ \pi$, which implies that $\eta(\text{Ker } \pi) \subseteq \text{Ker } \varphi_L = I_L$. To prove that I_L is generated by the elements of $\eta(\text{Ker } \pi)$ it suffices to show that the above generators for I_L are contained in the set $\eta(\text{Ker } \pi)$. Now Ker π contains

$$Y_1Y_2Y_3 - Y_4Y_5Y_7,\ Y_4Y_6 - Y_1Y_8,\ Y_5Y_8 - Y_2Y_9 \text{ and } Y_7Y_9 - Y_3Y_6,$$

which implies that $\eta(\text{Ker } \pi)$ contains the above generators for the ideal I_L. Hence we get the affine toric variety Spec $k[S]$ associated with the gap sequence L, which implies that L is Weierstrass by Corollary 4.9 in Komeda [**7**].

The Case (2). Set $a_1 = 7, a_2 = 8, a_3 = 9$ and $a_4 = 12$. Then the ideal I_L is generated by

$$X_1^3 - X_3X_4,\quad X_2^2 - X_1X_3,\quad X_3^3 - X_1X_2X_4,$$
$$X_4^2 - X_1X_2X_3 \quad \text{and} \quad X_1^2X_4 - X_2X_3^2.$$

Set $g_1 = X_1$, $g_2 = X_1$, $g_3 = X_1$, $g_4 = X_3$, $g_5 = X_2$, $g_6 = X_2$, $g_7 = X_3$, $g_8 = X_4$, $g_9 = X_3$ and $g_{10} = X_4$. Let S be the subsemigroup of $\mathbb{Z}^7$ generated by $b_1, b_2, \ldots, b_{10}$, where $b_i = e_i$ for $i = 1, \ldots, 7$, $b_8 = e_1 + e_2 + e_3 - e_4$, $b_9 = e_5 + e_6 - e_1$ and $b_{10} = e_4 + e_6 + e_7 - e_1 - e_2$. Then in the similar way to the above case we can show that S is saturated and that Spec $k[S]$ is the affine toric variety associated with the gap sequence L.

The Case (4). Set $a_1 = 7, a_2 = 8, a_3 = 10$ and $a_4 = 12$. Then the ideal I_L is generated by

$$X_1^4 - X_2^2X_4,\quad X_2^3 - X_1^2X_3,\quad X_3^2 - X_2X_4,\quad X_4^2 - X_1^2X_3,$$
$$X_1^2X_2 - X_3X_4 \quad \text{and} \quad X_1^2X_4 - X_2^2X_3.$$

Set

$$g_1 = X_1^2,\quad g_2 = X_1^2,\quad g_3 = X_2^2,\quad g_4 = X_2,$$
$$g_5 = X_3,\quad g_6 = X_4,\quad g_7 = X_3 \quad \text{and} \quad g_8 = X_4.$$

Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $b_1, b_2, \ldots, b_8$, where $b_i = e_i$ for $i = 1, \ldots, 5$, $b_6 = e_1 + e_2 - e_3$, $b_7 = e_3 + e_4 - e_1$ and $b_8 = e_3 + e_5 - e_1$. Then we can see that Spec $k[S]$ is the affine toric variety associated with L.

Lastly we construct an affine toric variety associated with the gap sequence (3). The way of its construction is slightly different from the above cases.

The Case (3). Set $a_1 = 7, a_2 = 8, a_3 = 10$ and $a_4 = 11$. Then we have relations:

$$(d_{41} + d'_1)a_1 = d_{13}a_3 + d_{14}a_4,\ d'_1 a_1 + (d_{13} + d_{23})a_3 = d'_2 a_2 + d_{14}a_4,$$
$$2d_{14}a_4 = d_{41}a_1 + d_{42}a_2,\ (d_{42} + d'_2)a_2 = 2d'_1 a_1 + d_{23}a_3,$$
$$(2d_{13} + d_{23})a_3 = d_{41}a_1 + d'_2 a_2 \text{ and}$$
$$d'_1 a_1 + d_{14}a_4 = d_{42}a_2 + d_{13}a_3,$$

where we set $d_{41} = d'_2 = 2$ and $d'_1 = d_{13} = d_{14} = d_{23} = d_{42} = 1$. Hence the ideal I_L is generated by

$$X_1^{d_{41}+d'_1} - X_3^{d_{13}}X_4^{d_{14}},\ X_1^{d'_1}X_3^{d_{13}+d_{23}} - X_2^{d'_2}X_4^{d_{14}},$$
$$X_4^{2d_{14}} - X_1^{d_{41}}X_2^{d_{42}}, X_2^{d_{42}+d'_2} - X_1^{2d'_1}X_3^{d_{23}},$$
$$X_3^{2d_{13}+d_{23}} - X_1^{d_{41}}X_2^{d'_2} \quad \text{and} \quad X_1^{d'_1}X_4^{d_{14}} - X_2^{d_{42}}X_3^{d_{13}}.$$

Set

$$g_1 = X_1^{d_{41}}, \quad g_2 = X_1^{d'_1}, \quad g_3 = X_3^{d_{13}}, \quad g_4 = X_3^{d_{23}},$$
$$g_5 = X_4^{d_{14}}, \quad g_6 = X_2^{d'_2} \quad \text{and} \quad g_7 = X_2^{d_{42}}.$$

Let S be the subsemigroup of $\mathbb{Z}^4$ generated by $b_1, b_2, \ldots, b_7$, where $b_i = e_i$ for $i = 1, \ldots, 4$, $b_5 = e_1 + e_2 - e_3$, $b_6 = 2e_3 + e_4 - e_1$ and $b_7 = e_1 + 2e_2 - 2e_3$. Let

$$p = (p_1, \ldots, p_4) = \sum_{i=1}^{7} m_i b_i \in \mathbb{Z}^4$$

with $0 \leq m_i < 1$ for all i. Then $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq -2$ and $p_4 \geq 0$. Let $p_3 = -2$, i.e., $m_3 + 2m_6 + 2 = m_5 + 2m_7$. Then

$$p_1 = m_1 - m_6 + (m_3 + 2m_6 + 2 - m_7) = m_1 + m_3 + m_6 - m_7 + 2 \geq 2$$

and $p_2 = m_2 + m_3 + 2m_6 + 2 \geq 2$. Hence we may assume that $p = (2, 2, -2, 0)$, which implies that $p = b_1 + b_7 \in S$. Let $p_3 = -1$, i.e., $m_3 + 2m_6 + 1 = m_5 + 2m_7$. Then $p_1 \geq 1$ and $p_2 \geq 1$. Hence we may assume that $p = (1, 1, -1, 0)$, which implies that $p = b_5 \in S$. Therefore the semigroup S is saturated. Hence we get the affine toric variety

Spec $k[S]$ associated with the gap sequence L.

4. Dimensionally proper gap sequences.

In this section we show that the gap sequences (5) and (7) are Weierstrass. In fact, we can prove that these gap sequences satisfy the following:

Definition 4.1. For a gap sequence L of genus g, we define a locally closed subset of $\mathcal{M}_{g,1}$ by

$$\mathcal{C}_L = \{(C, P) \in \mathcal{M}_{g,1} \mid L(P) = L\},$$

where $\mathcal{M}_{g,1}$ denotes the moduli space of pointed curves of genus g. Then the weight $w(L)$ of L gives an upper bound for the codimension of any irreducible component of $\mathcal{C}_L$ in $\mathcal{M}_{g,1}$. The gap sequence L is said to be *dimensionally proper* if there exists an irreducible component of $\mathcal{C}_L$ of codimension $w(L)$, i.e., dimension $3g - 2 - w(L)$.

Using the theory of limit linear series Eisenbud-Harris [**4**] showed the following which is useful for investigating whether a primitive gap sequence is dimensionally proper.

Remark 4.2. Let L be a dimensionally proper gap sequence of genus $g-1$ with $\alpha(L) = (\alpha_0, \alpha_1, \ldots, \alpha_{g-2})$. Then the gap sequence M with $\alpha(M) = (\beta_0, \beta_1, \ldots, \beta_{g-1})$ is dimensionally proper if it satisfies one of the following:

1) $\beta_0 = 0$, $\beta_i = \alpha_{i-1}$ $(i = 1, \ldots, g-1)$,
2) for some $0 < j \leq g-1$, $\beta_0 = 0$, $\beta_j = \alpha_{j-1} + 1$, $\beta_i = \alpha_{i-1}$ $(i = 1, \ldots, g-1, i \neq j)$.

The Case (5)., i.e., $\alpha(L) = (0^6, 1, 4, 4)$. By Proposition 4.4 in Komeda [**10**] the gap sequence L_0 with $\alpha(L_0) = (0^6, 4, 4)$ is dimensionally proper. Hence it follows from Remark 4.2 that L is also dimensionally proper.

For the sequence (7) we use the following which is the main theorem in Stöhr-Viana [**14**].

Remark 4.3. Let g, σ and ρ be integers satisfying $g \geq 5$ and $\sigma < g < \rho < 2\sigma+2$. If $\rho \leq 3\left[\frac{g+1}{2}\right] + 4 - \sigma$, then the moduli space of pointed trigonal curves of genus g with gap sequence $\{1,\dots,\sigma,\sigma+\rho-g+1,\dots,\rho\}$ has dimension $2g+3-\rho+\sigma$.

In order to show using Remark 4.3 that the sequence (7) is dimensionally proper, we need the following remark which is due to Oliveira [**12**].

Remark 4.4. If (C,P) is a pointed curve of genus $g \geq 5$ with gap sequence $\{1,\dots,g-2,2g-4,2g-3\}$, then C is trigonal.

To see the truth of the above remark, calculate the dimension of the complete linear system $|K_C(-(2g-5)P)|$ where K_C is a canonical divisor on C.

The Case (7)., i.e., $\alpha(L)=(0^7,4,6)$. It follows from Remarks 4.3 and 4.4 that the gap sequence $L_1=\{1,2,3,4,5,10,11\}$ is dimensionally proper. Since we have $\alpha(L_1)=(0^5,4,4)$, using Remark 4.2 twice we see that L is dimensionally proper.

5. The moduli space of pointed curves with gap sequence $\{1,\dots,6,12,13\}$.

In the last section we shall show that the gap sequence (6), i.e., $L=\{1,\dots,7,13,15\}$, is Weierstrass. Since we have $\alpha(L)=(0^7,5,6)$, by Remark 4.2 it suffices to show that the gap sequence L_0 with $\alpha(L_0)=(0^6,5,5)$ is dimensionally proper. To calculate the dimension of $\mathcal{C}_{L_0}$ we prepare some notations and statements from Stöhr-Viana [**14**].

Definition 5.1. Let C be a trigonal curve of genus $g \geq 5$ and g^1_3 a unique trigonal linear system on C. For any positive integer i, set

$$\alpha_i = h^0((i+1)g^1_3) - h^0(ig^1_3).$$

Let

$$m = \mathrm{Min}\{i|\alpha_i \geq 2\} - 1 \quad \text{and} \quad n = \mathrm{Min}\{i|\alpha_i \geq 3\} - 1.$$

The integers m and n are called the *Maroni invariants* of C, which satisfy

$$m \leq n,\ g = m+n+2 \quad \text{and} \quad m \geq \frac{g-4}{3}.$$

(See page 252 in Coppens [**2**]).

Hereafter we are in the following situation: Let C be a trigonal curve of genus $g \geq 5$ with Maroni invariants m and n. Then we have a canonical embedding of C in the projective space $\mathbf{P}^{g-1}(k) = \mathbf{P}^{m+n+1}(k)$ and by choosing projective coordinates in a convenient way, we may assume that C lies on the rational normal scroll S_{mn} defined by the set

$$\{(x_0 : x_1 : \ldots : x_{m+n+1}) \in \mathbf{P}^{m+n+1}(k) \mid \\ \operatorname{rank}\begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{n+m} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_{n+m+1} \end{pmatrix} < 2\}.$$

Moreover, we define two nonsingular rational curves D and E which are contained in S_{mn} as follows:

$$D = \{(a^n : a^{n-1}b : \ldots : b^n : 0 : \ldots : 0) \mid (a : b) \in \mathbf{P}^1(k)\}$$

and

$$E = \{(0 : \ldots : 0 : a^m : a^{m-1}b : \ldots : b^m) \mid (a : b) \in \mathbf{P}^1(k)\}.$$

Let P be a point of C. Set $h_P = \max\{(C.B)_P \mid B \in |D|\}$ where $(C.B)_P$ denotes the intersection multiplicity of the curves C and B at the point P. Then h_P is an invariant of the pointed curve (C, P). (See page 70 in Stöhr-Viana [**14**]). Moreover, we call P an *exceptional point* if $m < n$ and if it lies on the curve E of negative self-intersection number $m - n$ on the ambient scroll.

Remark 5.2. If P is an *unramified point* of C, that is to say it is unramified over the trigonal covering $\pi : C \longrightarrow \mathbf{P}^1$, then

$$n - m < h_P \leq \begin{cases} 2n - m + 2 & \text{when } P \notin E, \\ m + 2 & \text{when } P \in E. \end{cases}$$

(See Corollary 2.3 in Stöhr-Viana [**14**]).

Remark 5.3. If P is an unramified point of C, then the integers $1, \ldots, n+1$ and $h_P + 1, \ldots, h_P + 1 + m$ are contained in $L(P)$. (See Proposition 2.4 in Stöhr-Viana [**14**]).

The following are the key propositions in Stöhr-Viana [**14**].

Remark 5.4. (1) Let h, s and r be integers satisfying

$$n - m < h \leq n - m + 1 + s \text{ and } 0 \leq s < m < r \leq n + 3 + 2s.$$

Then the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants m and n and of unramified nonexceptional points $P \in C$ with invariant $h_P = h$ and gap sequence $\{1, \ldots, n+2+s, n+3+s+r-m, \ldots, n+2+r\}$ form a quasi-projective rational algebraic variety of dimension $2g+5-h-r+s$ (resp. $2g+4-h-r+s$) when $m < n$ (resp. $m = n$), provided that

$$r \leq 3h+3m-2n-s \text{ and } h \leq m+3, \text{ or that } r \leq 2h+2m-n-s.$$

(See Proposition 3.4 (a) in Stöhr-Viana [**14**]).

(2) Let t, s and r be integers satisfying

$$1 \leq t \leq 2m-n+2 \text{ and } t-1 \leq s < m < r \leq n+3+2s.$$

Then the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants m and n and of unramified exceptional points $P \in C$ with invariant $h_P = h = n-m+t$ and gap sequence $\{1, \ldots, n+2+s, n+3+s+r-m, \ldots, n+2+r\}$ form a quasi-projective rational algebraic variety of dimension $2g+4-h-r+s$ provided that $r \leq 3t+n-s$. (See Proposition 3.5 (a) in Stöhr-Viana [**14**]).

In the case $L(P) = L_0$ the point P must be unramified (See Coppens [**2**], [**3**] and Kato-Horiuchi [**5**]). Hence we can calculate the Maroni invariants m and n and the invariant h_P of the pointed curve (C, P) as follows.

Lemma 5.5. *Let (C, P) be a pointed trigonal curve of genus 8 with $L(P) = L_0$ and Maroni invariants m and n. Then the following statements hold.*

(1) $(m, n) = (2, 4)$ *or* $(3, 3)$.

(2) *If* $(m, n) = (2, 4)$, *then* $h_P = 3$.

(3) *If* $(m, n) = (3, 3)$, *then* $h_P = 1$ *or* 2.

Applying Remark 5.4 to our case we get the following results on the dimensions of some subvarieties of the moduli space $\mathcal{C}_{L_0}$.

Proposition 5.6. (1) *The algebraic variety of the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants 2 and 4* *and of unramified nonexceptional (resp. exceptional) points $P \in C$ with invariant $h_P = 3$ and gap sequence L_0 has dimension* 11 *(resp.* 10*).*

(2) *The algebraic variety of the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants* 3 *and* 3 *and of unramified points $P \in C$ with invariant $h_P = 2$ and gap sequence L_0 has dimension* 11.

To calculate the dimension of C_{L_0}, by Lemma 5.5 and Proposition 5.6 it suffices to consider the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants 3 and 3 and of unramified points $P \in C$ with invariant $h_P = 1$ and gap sequence L_0.

Let (C, P) be a pointed curve as in the above. Then we may assume that the curve C is defined by the equation

$$\begin{aligned} 0 = f(x,y) = x &+ c_{20}x^2 + c_{30}x^3 + c_{40}x^4 + c_{50}x^5 \\ &+ (1 + c_{21}x^2 + c_{31}x^3 + c_{41}x^4 + c_{51}x^5)y \\ &+ (c_{12}x + c_{22}x^2 + c_{32}x^3 + c_{42}x^4 + c_{52}x^5)y^2 \\ &+ (c_{03} + c_{13}x + c_{23}x^2 + c_{33}x^3 + c_{43}x^4 + c_{53}x^5)y^3 \end{aligned}$$

and that the point P corresponds to $(x, y) = (0, 0)$. (See Theorem 1.1 and Proposition 3.1 (i) in Stöhr-Viana [**14**]). The isomorphism class of (C, P) determines the coefficients c_{ij} 's uniquely up to the substitution $c_{ij} \longrightarrow c^{i+j-1}c_{ij}$ where $c \in k^*$. Thus we attach to each c_{ij} the weight $i + j - 1$. Since P is unramified, x is a local parameter at P. We write y as a power series in the local parameter x, say $y = \sum_{l=1}^{\infty} b_l x^l$. Moreover, the gap sequence at P is equal to $1, 2, 3, 4, 5, 6, 12, 13$ if and only if $b_1 b_3 - b_2^2 \neq 0$ and

$$\frac{1}{b_1 b_3 - b_2^2}((b_3^2 - b_2 b_4)b_{l-2} + (b_1 b_4 - b_2 b_3)b_{l-1}) \begin{cases} = b_l & \text{when } l = 5,6,7,8,9,10, \\ \neq b_l & \text{when } l = 11 \end{cases}$$

(See Remark 2.8 in Stöhr-Viana [**14**]). Now we have

$$0 = f(x,y) = f(x, \sum_{l=1}^{\infty} b_l x^l).$$

Comparing the coefficients of x^r for each r (with $1 \leq r \leq 10$), we can write each b_r as a polynomial expression of the coefficients c_{ij} of the equation $f(x, y) = 0$ defining C. By using the relations among $b_1, b_2, b_3, b_4, b_{l-2}, b_{l-1}, b_l$ for each l (with $5 \leq l \leq 9$) we can show that the coefficients $c_{50}, c_{51}, c_{52}, c_{53}$ and c_{43} are written by rational expressions of the remaining 14 coefficients $c_{20}, c_{30}, c_{21}, c_{12}, c_{03}, c_{40}, c_{31}, c_{22}, c_{13}, c_{41}, c_{32}, c_{23}, c_{42}$ and c_{33}; the denominators are powers of $b_1 b_3 - b_2^2 = c_{30} - c_{21} + c_{12} - c_{03} - c_{20}^2$. Using the relation with $l = 10$ we obtain a non-trivial (and even irreducible) polynomial equation (of degree one in c_{42}) between the above 14 coefficients. Thus we get the following proposition:

Proposition 5.7. *The algebraic variety of the isomorphism classes of the pairs (C, P) of trigonal curves C with Maroni invariants* 3 *and* 3 *and of unramified points $P \in C$ with invariant $h_P = 1$ and gap sequence L_0 has dimension less than* 13.

Theorem 5.8. *We have* $\dim C_{L_0} = 12$. *Hence L_0 is dimensionally proper.*

References

[**1**] Buchweitz, R. O., *On Zariski's criterion for equisingularity and non-smoothable monomial curves.* preprint 113, University of Hannover, (1980).

[**2**] Coppens, M., *The Weierstrass gap sequences of the total ramification points of trigonal coverings of* $\mathbf{P}^1$. Indag. Math. **47:** (1985), 245-276.

[**3**] Coppens, M., *The Weierstrass gap sequences of the ordinary ramification points of trigonal coverings of* $\mathbf{P}^1$ *: Existence of a kind of Weierstrass gap sequence.* J. Pure Appl. Algebra. **43:** (1986), 11-25.

[**4**] Eisenbud, D. and Harris, J., *Existence, decomposition, and limits of certain Weierstrass points.* Invent. Math. **87:** (1987), 495-515.

[**5**] Kato, T. and Horiuchi, R., *Weierstrass gap sequences at the ramification points of trigonal Riemann surfaces.* J. Pure Appl. Algebra. **50:** (1988), 271-285.

[**6**] Kim, S.J., *On the existence of Weierstrass gap sequences on trigonal curves.* J. Pure Appl. Algebra. **63:** (1990), 171-180.

[**7**] Komeda, J., *On Weierstrass points whose first non-gaps are four.* J. Reine Angew. Math. **341:** (1983), 68-86.

[**8**] Komeda, J., *On primitive Schubert indices of genus g and weight $g - 1$.* J. Math. Soc. Japan. **43:** (1991), 437-445.

[**9**] Komeda, J., *On the existence of Weierstrass points whose first non-gaps are five.* Manuscripta Math. **76:** (1992), 193-211.

[**10**] Komeda, J., *On the existence of Weierstrass gap sequences on curves of genus* ≤ 8. J. Pure Appl. Algebra. **97:** (1994), 51-71.

[**11**] Maclachlan, C., *Weierstrass points on compact Riemann surfaces.* J. London Math. Soc. **3:** (1971), 722-724.

[**12**] Oliveira, G., *Weierstrass semigroups and the canonical ideals of non-trigonal curves.* Manuscripta Math. **71:** (1991), 431-450.

[**13**] Pinkham, H., *Deformations of algebraic varieties with G_m-action.* Astérisque **20:** (1974), 1-131.

[**14**] Stöhr, K.O. and Viana, P., *Weierstrass gap sequences and moduli varieties of trigonal curves.* J. Pure Appl. Algebra. **81:** (1992), 63-82.

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